A proof of the J-Generalization of the Göllnitz-Gordon-Andrews Identities via Commutative Algebra

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Based on joint work with Rupam Barman and Alapan Ghosh.

Outline

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1 Partition Identities

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3 Proof of the J-Generalization of the Göllnitz-Gordon-Andrews Identities

4 References

Partition

Definition 1 (Partition of a positive integer).

A partition of a positive integer n is a finite non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

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The partitions of 5 are:

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Thus, $p(5) = 7$.

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where $(a;q)_m = \prod_{n=1}^{m}(1-aq^{n-1})$, with $m \in \mathbb{N} \cup \{\infty\}$.

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$$= \sum_{n=1}^{\infty} p(n) q^n.$$

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- For detailed list of partition identities, see, Chapter 1 (p. 13) of

  ☛ G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

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Theorem 5 (Euler (1741)).

For all $n \geq 0$,

$$p_o(n) = p_d(n).$$

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&= \sum_{n=0}^{\infty} p_o(n)q^n.
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  For any $\lambda \in \mathcal{P}_d(n)$, if there is an even part in λ, say m, then write

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Example 6.

$\Psi_6 : \mathcal{P}_d(6) \to \mathcal{P}_o(6)$:

$$\begin{aligned} 6 &= 2 \cdot 3 && \mapsto 3 + 3 \\ 5 + 1 &= 5 + 1 && \mapsto 5 + 1 \\ 4 + 2 &= 2^2 \cdot 1 + 2 \cdot 1 && \mapsto 1 + 1 + 1 + 1 + 1 + 1 \\ 3 + 2 + 1 &= 3 + 2 \cdot 1 + 1 && \mapsto 3 + 1 + 1 + 1. \end{aligned}$$

A proof of Euler's Partition Identity-II

- Any partition can be written as

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + f_5 \cdot 5 + \cdots ,$$

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  For any $f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots \in \mathcal{P}_o(n)$, replace f_i with its binary expansion $a_{i,0} \cdot 1 + a_{i,1} \cdot 2 + a_{i,2} \cdot 4 + a_{i,3} \cdot 8 + \cdots$ $(a_{r,s} \in \{0, 1\})$:

$$\begin{aligned} & f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots \\ = & (a_{1,0} \cdot 1 + a_{1,1} \cdot 2 + a_{1,2} \cdot 4 + a_{1,3} \cdot 8 + \cdots) \cdot 1 \\ & + (a_{2,0} \cdot 1 + a_{2,1} \cdot 2 + a_{2,2} \cdot 4 + a_{2,3} \cdot 8 + \cdots) \cdot 3 \\ & + (a_{3,0} \cdot 1 + a_{3,1} \cdot 2 + a_{3,2} \cdot 4 + a_{3,3} \cdot 8 + \cdots) \cdot 5 + \cdots \\ = & a_{1,0} + 2a_{1,1} + 3a_{2,0} + 4a_{1,2} + 5a_{5,0} + 6a_{3,1} + 7a_{7,0} + \cdots \in \mathcal{P}_d(n). \end{aligned}$$

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- Therefore, $|\mathcal{P}_o(n)| \leq |\mathcal{P}_d(n)|$.

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$$\begin{array}{rcl} 5{+}1 &=& 1{\cdot}1 + 1\cdot 5 & \mapsto & (2^0)\cdot 1 + (2^0)\cdot 5 &=& 1{+}5 \\ 3{+}3 &=& 2\cdot 3 & \mapsto & (2^1)\cdot 3 &=& 6 \\ 3{+}1{+}1{+}1 &=& 3\cdot 1 + 1\cdot 3 & \mapsto & (2^0+2^1)\cdot 1 + (2^0)\cdot 3 &=& 1{+}2{+}3 \\ 1{+}1{+}1{+}1{+}1{+}1 &=& 6\cdot 1 & \mapsto & (2^1+2^2)\cdot 1 &=& 2{+}4. \end{array}$$

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- Hence, $|\mathcal{P}_o(n)| = |\mathcal{P}_d(n)|$, which implies $p_o(n) = p_d(n)$.

Glaisher's Partition Identity

Theorem 8 (Glaisher).

Let $k \geq 2$ be an integer. Let $p_{\nmid k}(n)$ denote the number of partitions of n with parts not divisible by k and let $p_{<k}(n)$ denote the number of partitions of n where no part appears d times or more. Then

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- Proof I:

$$\sum_{n=0}^{\infty} p_{<k}(n)q^n = \prod_{n\geq 1}(1 + q^n + \cdots + q^{(k-1)n}) = \prod_{n\geq 1} \frac{1-q^{kn}}{1-q^n}$$

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- Proof II: Glaisher's bijection.

Rogers-Ramanujan Identities

$$\text{(I)} \qquad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

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- Rogers (1894) proved these identities.

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The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to $\pm 1 \pmod 5$.

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Theorem 10 (2nd Rogers-Ramanujan Identity).

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Andrews-Gordon Identities

Theorem 11 (Gordon, Amer. J. Math. (1961)).

The number of partitions of n into parts not congruent to $0, \ \pm i \pmod{2r+1}$, *where* $1 \leq i \leq r$, *is equal to the number of partitions* $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ *of n such that* $\lambda_m - \lambda_{m+r-1} \geq 2$, *and* $\lambda_{s-i+1} \geq 2$.

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Theorem 12 (Andrews, Proc. Nat. Acad. Sci. USA (1974)).

let $r \ge 2$ and $1 \le i \le r$ be two integers. We have

$$\sum_{n_1 \ge \cdots \ge n_{r-1} \ge 0} \frac{q^{n_1^2+\cdots+n_{r-1}^2+n_i+\cdots+n_{r-1}}}{(q;q)_{n_1-n_r}\cdots(q;q)_{n_{r-2}-n_{r-1}}(q;q)_{n_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q;q)_\infty}.$$

Here, we use the notation $(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty$.

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Theorem 13 (1st Göllnitz-Gordon identity).

The number of partitions of any positive integer n into parts congruent to $1, 4, 7 \pmod 8$ is equal to the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n such that $\lambda_m - \lambda_{m+1} \geq 2$, and $\lambda_m - \lambda_{m+1} \geq 3$ if λ_m is even.

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Theorem 14 (2nd Göllnitz-Gordon identity).

The number of partitions of any positive integer n into parts congruent to $3, 4, 5 \pmod 8$ is equal to the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n such that $\lambda_s \geq 3$, $\lambda_m - \lambda_{m+1} \geq 2$, and $\lambda_m - \lambda_{m+1} \geq 3$ if λ_m is even.

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- Let $D_{r,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:
  1. No odd part is repeated,
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- The case $r = 2$ in Theorem 15 gives the GG identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. We take $r \geq 2$.

Rogers-Ramanujan type Identities

Theorem 16 (Wang-Wang, arXiv:2402.06253v1).

We have

$$\sum_{i,j,k\geq 0} \frac{q^{\frac{1}{2}i^2+j^2+4k^2+ij+2jk+\frac{1}{2}i+j}}{(q;q)_i(q;q)_j(q^2;q^2)_k} = \frac{(q^8,q^{12},q^{20};q^{20})_\infty}{(q;q)_\infty},$$

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Outline

Graded Ring and Graded Algebra

Definition 17 (Graded ring).

A graded ring is a ring A together with a family $(A_j)_{j\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{j=0}^{\infty} A_j$ and $A_{j_1}A_{j_2} \subseteq A_{j_1+j_2}$ for all $j_1, j_2 \geq 0$.

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Definition 19 (Graded algebra).

Let $\mathbb{F}$ be a field. A graded ring $A = \bigoplus_{j=0}^{\infty} A_j$ is called a graded $\mathbb{F}$-algebra if it is also an $\mathbb{F}$-algebra, and A_j is a vector space for all $j \geq 0$ with $A_0 = \mathbb{F}$.

Homogeneous ideal

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Remark 21.

1. *A homogeneous ideal I is the direct sum of its homogeneous parts $I_j = I \cap A_j$, i.e., $I = \bigoplus_{j=0}^{\infty} I_j$.*
2. *If I is a homogeneous ideal of a graded ring A, then the quotient ring $\frac{A}{I}$ is also a graded ring, decomposed as*

$$\frac{A}{I} = \bigoplus_{j=0}^{\infty} \frac{A_j}{I_j}.$$

Hilbert-Poincaré series

Definition 22 (Hilbert-Poincaré series).

Let $\mathbb{F}$ be a field of characteristic zero and $A = \bigoplus_{j=0}^{\infty} A_j$ be a graded $\mathbb{F}$-algebra such that $\dim_{\mathbb{F}}(A_j) < \infty$. Then the Hilbert-Poincaré series of A is

$$\mathrm{HP}_A(q) := \sum_{j \geq 0} \dim_{\mathbb{F}}(A_j) q^j.$$

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Definition 23 (Weight of a polynomial).

The weight of the monomial $x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m} \in \mathbb{F}[x_1, x_2, \ldots]$ is defined as $\sum_{k=1}^{m} i_k \alpha_k$. A polynomial $f(x) \in \mathbb{F}[x_1, x_2, \ldots]$ is said to be a homogeneous polynomial of weight a if every monomial of $f(x)$ has the same weight a.

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Example 24 (Gradation by weight).

$S := \mathbb{F}[x_1, x_2, \ldots]$ is a graded by weight, i.e., $S = \bigoplus_{j=0}^{\infty} (S)_j$, where $(S)_j$ is the set of polynomials of weight j along with zero polynomial.

An Example

Example 25.

The Hilbert-Poincaré series of $S := \mathbb{F}[x_1, x_2, \ldots]$ is equal to the generating function of the partiiton function $p(n)$. That is

$$\mathrm{HP}_S(q) = \sum_{j\geq 0} \dim_{\mathbb{F}}((S)_j) q^j = \sum_{j=0}^{\infty} p(j) q^j = \prod_{n\geq 1} \frac{1}{1-q^n}.$$

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$$\mathrm{HP}_S(q) = \sum_{j \geq 0} \dim_{\mathbb{F}}((S)_j) q^j = \sum_{j=0}^{\infty} p(j) q^j = \prod_{n \geq 1} \frac{1}{1 - q^n}.$$

As there is a bijection between

$$T_{(S)_j} := \left\{ x_{l_1} x_{l_2} \cdots x_{l_m} \in S \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j \right\}$$

and

$$T_{p(j)} := \left\{ (l_1, l_2, \ldots, l_m) \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j \right\}.$$

An Example

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Therefore,

$$\dim_{\mathbb{F}}((S)_j) = |T_{(S)_j}| = |T_{p(j)}| = p(j).$$

Outline

Generating function of $C_{r,i}(n)$

- For $1 \leq i \leq r$, define

$$C_i(q) := \prod_{\substack{m \geq 1,\ m \not\equiv 2 \pmod 4 \\ m \not\equiv 0,\ 2r \pm (2i-1) \pmod{4r}}} \frac{1}{1-q^m}.$$

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- For $1 \leq i \leq r$,

$$C_{r-i+1}(q) = \prod_{\substack{m\geq 1,\ m\not\equiv 2 \pmod 4 \\ m\not\equiv 0,\ \pm(2i-1) \pmod{4r}}} \frac{1}{1-q^m}.$$

is the generating function of $C_{r,i}(n)$.

J-Generalization of $C_i(q)$

For $J \geq 1$, Coulson et al. [Ramanujan J. (2017)] defined:
For $i = 1$

$$C_{(r-1)J+1}(q) = C_{(r-1)(J-1)+r}(q)$$

and for $i = 2, \ldots, r$

$$C_{(r-1)J+i}(q) = \frac{C_{(r-1)(J-1)+r-i+1}(q) - C_{(r-1)(J-1)+r-i+2}(q)}{q^{2J(i-1)}} - \frac{C_{(r-1)J+i-1}(q)}{q}.$$

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

- Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$.

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

- Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$.
- Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:
  1. No odd part is repeated,
  2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
  3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even,
  4. all parts are greater than $2J$, and
  5. at most $i - 1$ parts are equal to $2J + 1$ or $2J + 2$.

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- Let $\mathcal{E}_{r,i,J}(q)$ denote the generating function of $E_{r,i,J}(n)$.

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- Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$.
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- Let $\mathcal{E}_{r,i,J}(q)$ denote the generating function of $E_{r,i,J}(n)$.
- We note that $E_{r,i,0}(n) = D_{r,i}(n)$.

J-generalization of GGA Identities

The following theorem gives more general identities than the Göllnitz-Gordon-Andrews identities.

Theorem 26 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J and $1 \le i \le r$, we have

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

where $\ell = r - i + 1$.

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- The case $J = 0$ in Theorem 26 gives the Göllnitz-Gordon-Andrews identities.
- It is not known if $C_{(r-1)J+\ell}(q)$ for $J \geq 1$ is generating function of some partition function.

The ideal $L_{r,i,J}$

Let $a,\ b,\ c,\ n_1$, and n_2 be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$L_{r,i,J} := \big(x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^{i},\ x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : 2a-1, 2b-1, 2c \geq 2J+2; 0 \leq n_1 \leq r-1; 0 \leq n_2 \leq r-2 \big).$$

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$$\dim_{\mathbb{F}}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j = \dim_{\mathbb{F}}\left(\frac{(S_{2J+1})_j}{(L_{r,i,J})_j}\right) \leq \dim_{\mathbb{F}}((S_{2J+1})_j) \leq \dim_{\mathbb{F}}((S)_j) = p(j) < \infty.$$

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This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$:

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- $L_{r,i,J}$ is generated by x_{2J+1}^2, $x_{2J+1}x_{2J+2}^{i-1}$, x_{2J+2}^i and the monomials of the form

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  such that $2a-1, 2b-1, 2c \geq 2J+2$, $0 \leq n_1 \leq r-1$, and $0 \leq n_2 \leq r-2$.

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such that $2a-1, 2b-1, 2c \geq 2J+2$, $0 \leq n_1 \leq r-1$, and $0 \leq n_2 \leq r-2$.
- $\dim_{\mathbb{F}} \left(\frac{S_{2J+1}}{L_{r,i,J}} \right)_j = E_{r,i,J}(j)$.

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This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$:

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- $\dim_{\mathbb{F}}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j = E_{r,i,J}(j)$.

Therefore,

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The ideals L_k and L_k^ℓ

Let $a,\ b,\ c,\ n_1,\ n_2$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq 2J+1$:

$$L_k := \Big(x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : \\ 2a-1, 2b-1, 2c \geq k,\ 0 \leq n_1 \leq r-1,\ \text{and } 0 \leq n_2 \leq r-2 \Big)$$

and

$$L_k^\ell := \begin{cases} \Big(x_k^\ell,\ x_k^{\ell-1}x_{k+2}^{r-\ell+1},\ x_k^{\ell-2}x_{k+2}^{r-\ell+2}, \ldots, x_k x_{k+2}^{r-1},\ x_k^{\ell-1}x_{k+1}x_{k+2}^{r-\ell}, \\ \quad x_k^{\ell-2}x_{k+1}x_{k+2}^{r-\ell+1}, \ldots,\ x_k x_{k+1} x_{k+2}^{r-2},\ L_{k+1} \Big) & \text{if } k \text{ is even;} \\ \Big(x_k^2,\ x_k x_{k+1}^{\ell-1},\ L_{k+1}^\ell \Big) & \text{if } k \text{ is odd.} \end{cases}$$

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- L_k and L_k^ℓ are homogeneous ideals of S_k.
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HP_i^{2J+1}

- We denote $\mathrm{HP}_{\frac{S_k}{L_k^\ell}}(q)$ by HP_ℓ^k.

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- Therefore, $\frac{S_{2J+1}}{L_{r,i,J}} = \frac{S_{2J+1}}{L_{2J+1}^i}$.

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- Therefore,

$$\mathrm{HP}_i^{2J+1} = \mathrm{HP}_{\frac{S_{2J+1}}{L_{2J+1}^i}}(q) = \mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \mathcal{E}_{r,i,J}(q).$$

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- Note that $L_{r,i,J} = L_{2J+1}^i$.
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- Therefore,

$$\mathrm{HP}_i^{2J+1} = \mathrm{HP}_{\frac{S_{2J+1}}{L_{2J+1}^i}}(q) = \mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \mathcal{E}_{r,i,J}(q).$$

To prove

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

it is enough to prove that

$$C_{(r-1)J+\ell}(q) = \mathrm{HP}_i^{2J+1}$$

for any nonnegative integer J and $1 \le i \le r$, where $\ell = r - i + 1$.

General recursion formula for HP_ℓ^k

Lemma 27 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer. Let k, r, and ℓ be positive integers with $r \geq 2$ and $1 \leq \ell \leq r$. Then for any odd $k \geq 2J+1$, we have

$$\mathrm{HP}_\ell^k = \sum_{j=1}^{\ell-1} q^{(k+1)j-1} \mathrm{HP}_{r-j+1}^{k+2} + \sum_{j=1}^{\ell} q^{(k+1)(j-1)} \mathrm{HP}_{r-j+1}^{k+2}.$$

Recursion formula for HP_i^{2J+1}

Lemma 28 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{2J+1} = \sum_{j=1}^{r} N^J_{i,j,(r-1)d+j} \mathrm{HP}^{2d+1}_{r-j+1},$$

where $d \geq J+1$. Here, the coefficients $N^J_{i,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$N^J_{i,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} N^J_{i,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} N^J_{i,m,(r-1)d+m}$$

with the following initial conditions (for $d = J+1$)

$$N^J_{i,j,(r-1)(J+1)+j} = \begin{cases} q^{2(J+1)j-1} + q^{2(J+1)(j-1)} & \text{if } 1 \leq j \leq i-1; \\ q^{2(J+1)(j-1)} & \text{if } j = i; \\ 0 & \text{if } i+1 \leq j \leq r. \end{cases}$$

Recursion formula for $C_{(r-1)J+\ell}$

Lemma 29 (Coulson et al., Ramanujan J. (2017)).

Let J be a nonnegative integer and r, ℓ be integers with $r \geq 2$, $1 \leq \ell \leq r$. Then for $d \geq J+1$ we have the following recursion formula

$$C_{(r-1)J+\ell} = \sum_{j=1}^{r} M^J_{\ell,j,(r-1)d+j} C_{(r-1)d+j}.$$

Here, the coefficients $M^J_{\ell,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$M^J_{\ell,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} M^J_{\ell,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} M^J_{\ell,m,(r-1)d+m}$$

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$$M^J_{\ell,j,(r-1)(J+1)+j} = \begin{cases} q^{2(J+1)j-1} + q^{2(J+1)(j-1)} & \text{if } 1 \leq j \leq r-\ell; \\ q^{2(J+1)(j-1)} & \text{if } j = r-\ell+1; \\ 0 & \text{if } r-\ell+2 \leq j \leq r. \end{cases}$$

Equality of the coefficients $M^J_{\ell,v,(r-1)d+v}$ and $N^J_{i,v,(r-1)d+v}$

To prove that $\mathrm{HP}_i^{2J+1} = C_{(r-1)J+\ell}$, where $\ell = r - i + 1$, we prove that the coefficients in the recursion formulae of HP_i^{2J+1} (Lemma 28) and $C_{(r-1)J+\ell}$ (Lemma 29) are equal.

Lemma 30 (Barman-Ghosh-Singh (Submitted)).

For all $d \geq J + 1$, $r \geq 2$, and $1 \leq v \leq r$, we have

$$M^J_{\ell,v,(r-1)d+v} = N^J_{i,v,(r-1)d+v},$$

where $\ell = r - i + 1$.

Equality of the coefficients $M^J_{\ell,v,(r-1)d+v}$ and $N^J_{i,v,(r-1)d+v}$

To prove that $\mathrm{HP}_i^{2J+1} = C_{(r-1)J+\ell}$, where $\ell = r - i + 1$, we prove that the coefficients in the recursion formulae of HP_i^{2J+1} (Lemma 28) and $C_{(r-1)J+\ell}$ (Lemma 29) are equal.

Lemma 30 (Barman-Ghosh-Singh (Submitted)).

For all $d \geq J + 1$, $r \geq 2$, and $1 \leq v \leq r$, we have

$$M^J_{\ell,v,(r-1)d+v} = N^J_{i,v,(r-1)d+v},$$

where $\ell = r - i + 1$.

Theorem 31 (Barman-Ghosh-Singh (Submitted)).

For $r \geq 2$, $1 \leq i \leq r$, and $J \geq 0$, we have

$$C_{(r-1)J+\ell} = \mathrm{HP}_i^{2J+1},$$

where $\ell = r - i + 1$.

Outline

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Thank you